Recitation #5

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1 Annoucement

- Problem Set 2 is collected, Full mark is 5, one random question (other than question 4) is worth 2 points, the rest worth 1 point each
- Office hour for Midterm: **Friday, Oct 18, 2-4pm** and **Tuesday, Oct 22, 1-3pm**
- Email questions are welcomed, available 24/7 before Midterm
- Appointment is also strongly welcomed, available on Thursday before 4pm, Friday other than afternoon, and Wednesday 2-4pm

2 Question 1

Consider the utility function $u(x, y) = \sqrt{x} + \sqrt{y}$.

Problem 1 *Calculate the Marshallian Demand function for each good.*
Proof. *Since the utility function is monotonic and convex (Why?) and thus we can use the MRS method to solve for the utility maximization problem*

$$MRS = \frac{MU_x}{MU_y} = \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}} = \sqrt{\frac{y}{x}}$$

We know that at the optimal, we have

$$MRS = \frac{p_x}{p_y}$$

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This implies

$$\sqrt{\frac{y}{x}} = \frac{p_x}{p_y}$$
$$\Rightarrow \frac{y}{x} = \left(\frac{p_x}{p_y}\right)^2$$
$$\Rightarrow y = \left(\frac{p_x}{p_y}\right)^2 x \tag{1}$$

Now use (1) to plug into the budget constraint, we have

$$p_x x + p_y y = w$$
$$\Rightarrow p_x x + p_y \left(\frac{p_x}{p_y}\right)^2 x = w$$
$$\Rightarrow x^M(p_x, p_y, w) = \frac{w}{p_x + p_y\left(\frac{p_x}{p_y}\right)^2} = \frac{p_y w}{p_x(p_x + p_y)} \tag{2}$$

Now we have found the Mashallian demand function for good x, then you can use either budget constraint or (1) to back out the demand function for good y (I will use equation (1))

$$y^M(p_x, p_y, w) = \left(\frac{p_x}{p_y}\right)^2 x = \left(\frac{p_x}{p_y}\right)^2 \frac{p_y w}{p_x(p_x + p_y)} = \frac{p_x w}{p_y(p_x + p_y)} \tag{3}$$

Thus we find the Marshallian demand function as follows,

$$\begin{aligned} x^M(p_x, p_y, w) &= \frac{p_y w}{p_x(p_x + p_y)} \\ y^M(p_x, p_y, w) &= \frac{p_x w}{p_y(p_x + p_y)} \end{aligned}$$

Sai's Remark: *you should by now VERY familiar with the above method now for finding Marshalian demand function, this question is a fair game in the midterm or final, make sure to fully understand them.* ∎

Problem 2 *Calculate the income elasticity of demand for good x.*

Proof. By definition, the income elasticity of demand for good x,

$$\eta_x = \frac{\partial x}{\partial w}\frac{w}{x}$$

In which, we use the Marshallian demand we derived in problem 1, and solve for η_x

$$\begin{aligned} \eta_x &= \frac{\partial\left(\frac{p_y w}{p_x(p_x+p_y)}\right)}{\partial w}\frac{w}{\left(\frac{p_y w}{p_x(p_x+p_y)}\right)} \\ &= \frac{p_y}{p_x(p_x+p_y)}\frac{p_x(p_x+p_y)}{p_y} \\ &= 1 \end{aligned}$$

So we've found that the income elasticity of demand for good x is 1

Sai's Remark: η_x should be a function of prices and wealth, NOT x ■

Problem 3 *Is good x a normal good or inferior good? Luxury good or Necessity good?*

Proof. By the definition in the class, one good is called normal good if $\frac{\partial x}{\partial w} > 0$, otherwise it is inferior good

We have by Marshallian demand function that

$$\frac{\partial x}{\partial w} = \frac{w}{p_x(p_x+p_y)} > 0$$

Therefore, good x is a NORMAL good

In addition, by definition, one good is called luxury good if $\eta_x > 1$ and called necessity good if $\eta_x < 1$

But as shown in problem 2, $\eta_x = 1$, thus we conclude, good x is neither luxury good nor necessity good. ■

Problem 4 *Now if price of good x doubles, calculate the Compensating variation. In particular, suppose initial prices are $p_x = p_y = 1, w = 8$, and after the change $p'_x = 2$, $p'_y = 1$*

Proof. To be clear, our task is to find w', the wealth with which the agent is indifferent with his consumption before the price change

Let's first find x' and y'

MRS is NOT affected by the change of prices, but the price ratio IS affected accordingly, in particular, we have

$$
\begin{aligned}
MRS &= \frac{p'_x}{p'_y} \\
\Rightarrow \sqrt{\frac{y'}{x'}} &= 2 \\
\Rightarrow y' &= 4x' \qquad (4)
\end{aligned}
$$

Now plug (4) into budget constraint, and we get

$$
\begin{aligned}
p'_x x' + p'_y y' &= w' \\
\Rightarrow 2x' + 4x' &= w' \\
\Rightarrow x' &= \frac{w'}{6}
\end{aligned}
$$

Then you can plug in either budget constraint or (4) to back out y' (I use (4))

$$y' = 4x' = \frac{2w'}{3}$$

Now we plug our findings into utility function

$$u(x', y') = \sqrt{\frac{w'}{6}} + \sqrt{\frac{2w'}{3}}$$

Now we need to find the previous utility under the **original** price, by the result in problem 1, we have

$$
\begin{aligned}
x &= \frac{p_y w}{p_x(p_x + p_y)} = \frac{8}{2} = 4 \\
y &= \frac{p_x w}{p_y(p_x + p_y)} = \frac{8}{2} = 4
\end{aligned}
$$

Therefore, the **original** utility is

$$u(x, y) = \sqrt{4} + \sqrt{4} = 4$$

Therefore, since we need $u(x', y') = u(x, y)$, we can back out w'

$$
\begin{aligned}
u(x', y') &= u(x, y) \\
\Rightarrow \sqrt{\frac{w'}{6}} + \sqrt{\frac{2w'}{3}} &= 4 \\
\Rightarrow w' &= \frac{32}{3}
\end{aligned}
$$

Note In the derivation, I jumped some algbra steps, see Appendix for more detailed if you have confusion

Thus the Compentating Variation,

$$CV = w' - w = \frac{32}{3} - 8 = \frac{8}{3}$$

Sai's Remark: This is a general method to find CV when the utility is locally non-satiated and convex (so that we can use MRS = price ratio formula). When the utility is not monotonic or non-convex, you need to use Lagranging method to find x' and y', as functions of w'. Then use the indifference condition to solve for w' ■

Problem 5 *Calculate the Hicksian Demand Function of both goods*

Proof. We first set up the cost-minimization problem first, under environment (p_x, p_y, u)

$$
\begin{aligned}
\min \; & p_x x + p_y y \\
s.t \; & \sqrt{x} + \sqrt{y} \geq u
\end{aligned}
$$

To solve for Hicksian Deman function, we can use two methods[2]: Lagranging and MRS method. I will use the MRS method since the utility is monotonic and convex.

$$
\begin{aligned}
MRS &= \frac{p_x}{p_y} \\
\Rightarrow \sqrt{\frac{y}{x}} &= 1 \\
\Rightarrow y &= \left(\frac{p_x}{p_y}\right)^2 x \qquad (5)
\end{aligned}
$$

[2]Of course, you can always use substituion method to solve it, but it is way tedious

Then plug in the constraint in COST-MINIMIZATION problem (**WARNING**: when you calculate the Hicksian demand, the constraint is NOT budget constraint, instead, it is the constraint $\sqrt{x} + \sqrt{y} \geq u$)

Then plug (6) into constraint $\sqrt{x} + \sqrt{y} = u$, we have

$$\sqrt{x} + \sqrt{\left(\frac{p_x}{p_y}\right)^2 x} = u$$
$$\Rightarrow x^H(p_x, p_y, u) = \left(\frac{p_y u}{p_y + p_x}\right)^2$$

Now we can plug the $x^H(p_x, p_y, u)$ into either (6) or constraint $\sqrt{x} + \sqrt{y} = u$ to back out $y^H(p_x, p_y, u)$.I will use (6)

$$\begin{aligned} y^H(p_x, p_y, u) &= \left(\frac{p_x}{p_y}\right)^2 x^H(p_x, p_y, u) \\ &= \left(\frac{p_x}{p_y}\right)^2 \left(\frac{p_y u}{p_y + p_x}\right)^2 \\ &= \left(\frac{p_x u}{p_y + p_x}\right)^2 \end{aligned}$$

Thus we have solved our Hicksian Demand function

$$\begin{aligned} x^H(p_x, p_y, u) &= \left(\frac{p_y u}{p_y + p_x}\right)^2 \\ y^H(p_x, p_y, u) &= \left(\frac{p_x u}{p_y + p_x}\right)^2 \end{aligned}$$

Sai's Remark 1: remember we derived in problem 4, if $p_x = p_y = 1, \mathbf{w} = \mathbf{8}$, we have

$$\begin{aligned} x^M(p_x &= 1, p_y = 1, w = 8) = \frac{(1)8}{(1)(1+1)} = 4 \\ y^M(p_x &= 1, p_y = 1, w = 8) = 4 \qquad (6) \end{aligned}$$

And we also have

$$u(x^M, y^M) = \sqrt{4} + \sqrt{4} = 4 \qquad (7)$$

In particular, if $p_x = p_y = 1, \mathbf{u}^c\mathbf{= 4}$, we have

$$\begin{aligned} x^H(p_x &= 1, p_y = 1, u^c = 4) = \left(\frac{4}{2}\right)^2 = 4 \\ y^H(p_x &= 1, p_y = 1, u^c = 4) = \left(\frac{4}{2}\right)^2 = 4 \end{aligned} \tag{8}$$

And the total expediture function,

$$E(p_x = 1, p_y = 1, u^c = 4) = p_x x^H + p_y y^H = 8 \tag{9}$$

I am like: wait a minute, what?! From (6) an (8), we have

$$\begin{aligned} x^M(p_x &= 1, p_y = 1, w = 8) = x^H(p_x = 1, p_y = 1, u^c = 4) \\ y^M(p_x &= 1, p_y = 1, w = 8) = y^H(p_x = 1, p_y = 1, u^c = 4) \end{aligned}$$

and from (7) and (9) along with the environment $w = 8$ and $u^c = 4$, we observe that,

$$E(p_x = 1, p_y = 1, u = 4) = w$$
$$u(x^M, y^M) = u^c$$

In fact, it is NOT a surprice, we witness a very powerful result in Micro-Economics history, it's called **Duality:** if the preference is well behaved (monotonic, convex), then under same price, the solution to the utility maximization is also the solution to the cost-minimization. In particular, the Mashallian demand evalued at wealth of the minimized expenditure found in cost-minimization problem is EQUAL to the Hicksian demand evaluted at utility of maximized utility found in utility maximization problem. [3]

Sai's Remark 2. (WARNING). Both Hicksian demand and Expenditure function are functions in terms of prices and u (NOT in terms of x, y, w) ■

[3] Of course, unless professor covered this duality concept in class, you are not responsible for knowing how to derive it